[1] Fowler and Nordheim, *Proc. Roy. Soc.*, **A119,** 1 (1928).

[2] R. W. Gurney and E. U. Condon, *Phys. Rev.*, **33,** 127 (1929); Gamow, *Zeit. Phys.*, **51,** 204 (1928).

[3] Cf. C. N. Hinshelwood, *Chem. Rev.*, **3,** 227 (1926).

[4] Jeffries, *Lond. Math. Soc.*, p. 428, **1923.**

[5] Although the expression used for $P(W)$ is valid provided that W has one of the characteristic values, the results will not be far wrong if we employ the same form for a continuously varying W.

[6] O. K. Rice and H. C. Ramsperger, *J. A. C. S.*, **49,** 1617 (1927); O. K. Rice, *Proc. Nat. Acad. Sci.*, **14,** 113 (1928); N. S. Kassel, *Ibid.*, **14,** 23 (1928); R. H. Fowler, *Statistical Mechanics.*

A GENERALIZATION OF ROLLE'S THEOREM WITH APPLICATION TO ENTIRE FUNCTIONS

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1. Rolle's theorem for a function of a real variable $f(x)$ gives a sufficient condition for the existence of roots of the derivative in a certain interval, namely, between two consecutive zeros of $f(x)$. The object of the present note is to find a similar criterion for the existence and number of the roots of the derivative of a function of a complex variable $f(z)$, in the circle (C) drawn on the segment joining two zeros of $f(z)$ as a diameter, provided there is no other zero of $f(z)$ on or inside that circle.

2. We may assume, by a convenient choice of the origin, that the zeros of $f(z)$ are the points $\pm\lambda$ on the real axis. Further, we suppose that $f(z)$ is holomorphic in a region (D) containing (C). We have therefore

$$f(z) = (z^2 - \lambda^2)g(z),$$

$g(z)$ being holomorphic in (D) and without zeros in (C). The zeros of $f'(z)$ are given by the equation

$$(z^2 - \lambda^2)\frac{g'(z)}{g(z)} + 2z = 0. \qquad (1)$$

Let $g'(z)/g(z) = 1/\lambda u(z)$ and solve (1) as a quadratic equation in z; we have

$$z = \lambda\left[\pm(1 + u^2)^{1/2} - u\right] = F(z).$$

The number of roots of this equation in the circle (C) is given by Cauchy's integral

$$n = \frac{1}{2\pi i}\int_C \frac{F'(z) - 1}{F(z) - z}\,dz = \frac{1}{2\pi}\left\{\arg[F(z) - z]\right\}_C.$$

Let us consider first the simple case when the two determinations of $F(z)$ are uniform in (C), and, moreover, the region (Γ) corresponding to (C) in the transformation $Z = F(z)$ — $F(z)$ being one of the determinations—is inside (C). It follows immediately that

$$\{\arg[F(z) - z]\}_C = 2\pi$$

and therefore $$n = 1.$$

From the assumptions made on $F(z)$ it is very easy to obtain the corresponding conditions for $g(z)$, and we get finally the following result:

THEOREM 1.—*If, in the transformation* $Z = X + iY = g'(z)/g(z)$, *the circle* (C) *corresponds to a region* (Δ) *such that the relations* $X = 0$, $|Y| > 1/\lambda$ *are never simultaneously verified at a point of* (Δ), *there exists one and only one zero of* $f'(z)$ *in the circle* (C).

In other words, the region (Δ'), corresponding to (C) in the transformation $u = u(z)$, is contained in the plane of the variable u, cut by the segment L joining the points $u = +i$ and $u = -i$.

3. Let us consider next the case when the boundary of (Δ') cuts L at two points M and N, the segment MN being interior to (Δ'), and suppose, moreover, that the correspondence between (C) and (Δ), and therefore, between (C) and (Δ') is a (1, 1) correspondence. Let P and Q be the points of the circumference (C) corresponding to M and N, respectively, and PlQ the curve corresponding to MN.

Now the transformation

$$U = \lambda[(1 + u^2)^{1/2} - u]$$

with the proper determination (namely, $(1 + u^2)^{1/2} > 0$, when u is a real positive number), carries the u-plane cut by the segment L into the interior of (C); the right hand border of the cut L corresponding to the semi-circle of (C) to the right of the imaginary axis, and the left border of L to the other semi-circle of (C).

By dividing (Δ') into two regions (Δ_1') and (Δ_2') by means of MN, we thus manage to transform (C) into two regions (Γ_1) and (Γ_2) situated inside (C), by the transformation $Z = F(z)$, where the determination of $F(z)$ has been chosen as said above. These two regions have as parts of their boundary corresponding to MN, two arcs m_1n_1 and m_2n_2 of the circle (C) symmetrical with respect to the imaginary axis; if M' and N' are the points where m_1m_2 and n_1n_2 cut that axis, we have

$$\Im\{M'\} = -\lambda\,\Im\{M\}$$
$$\Im\{N'\} = -\lambda\Im\{N\}$$

The number of zeros of $f'(z)$ inside (C) is then given by

$$n = \frac{1}{2\pi i}\left[\int_{C_1} \frac{F'(z) - 1}{F(z) - z}\, dz + \int_{C_2} \frac{F'(z) - 1}{F(z) - z}\, dz\right],$$

(C_1) and (C_2) being the boundaries of the parts of (C) which correspond respectively to (Γ_1) and (Γ_2). We can always assume that $\Im\{M'\} < \Im\{N'\}$. A study of the above integrals leads to the following results:

$$\begin{array}{llllll} n = 2 & & \text{if} & \Im\{P\} > \Im\{M'\} & \text{and} & \Im\{Q\} < \Im\{N'\} \\ n = 1 & & \text{if} & \Im\{P\} > \Im\{M'\} & \text{and} & \Im\{Q\} > \Im\{N'\} \\ & \text{or} & \text{if} & \Im\{P\} < \Im\{M'\} & \text{and} & \Im\{Q\} < \Im\{N'\} \\ n = 0 & & \text{if} & \Im\{P\} < \Im\{M'\} & \text{and} & \Im\{Q\} > \Im\{N'\} \end{array}$$

which can be expressed in the form $n = 1 + \delta$, where

$$2\delta = \frac{\Im\{P\} - \Im\{M'\}}{|\Im\{P\} - \Im\{M'\}|} - \frac{\Im\{Q\} - \Im\{N'\}}{|\Im\{Q\} - \Im\{N'\}|}. \qquad (2)$$

We will refer to δ as the index of the cut PQ. A similar study may be carried out when the boundary of (Δ') cuts L in a single point, one of the points $\pm i$ being interior to (Δ').

4. If we consider finally the general case when the boundary of (Δ') cuts L in any number of points, the correspondence between (C) and (Δ') being (1,1), we can divide (Δ') by a suitable number of auxiliary cuts, into a finite number of regions, which either do not cut L, or are of one of the two kinds studied above. This gives, therefore, the following theorem.

THEOREM II.—*If the correspondence* $Z = g'(z)/g(z)$ *between the circle* (C) *and the region* (Δ) *is* (1,1), *the number of zeros of* $f'(z)$ *inside the circle* (C) *is given by*

$$n = 1 + \sum_k \delta_k, \qquad (3)$$

δ_k *being the index of the* k^{th} *cut of* (C), *each* δ_k *being given by a formula of the type* (2).

5. The restriction imposed upon the transformation $Z = g'(z)/g(z)$ can be very easily removed; the transformation $u = u(z)$ defines a certain Riemann surface, and the region (Δ') spreads over a finite number of sheets. If we draw a cut joining the points $+i$ and $-i$ in each one of these sheets, we can carry out the argument as above. There can be only one exception, when one of the points $\pm i$ is a branch-point of the surface; then several cuts will merge into a single one on the surface, which changes slightly the formula (2) giving the indices. The value of those indices can easily be found in each particular case, according to the way in which the sheets are connected at the branch-point.

6. The criterion we have thus obtained enables us to prove Rolle's theorem for analytic functions of a real variable. For the cuts of the circle (C) are then symmetrical to each other with respect to the real axis, and as no one can be symmetrical to itself, owing to the fact that $f(z)$ has no other zero in (C), their number is even. On the other hand, two symmetrical cuts have the same index. Therefore, n is an odd number, which proves that there is at least one zero of $f'(z)$ on the real axis between the points $\pm\lambda$.

7. Another immediate application is to Laguerre's theorem for entire functions of genus zero and one, of the type

$$f(z) = e^{kz}\,\Pi\left(1 - \frac{z}{a_n}\right)e^{\frac{z}{a_n}}, \tag{4}$$

where k and a_n are real numbers such that $\sum 1/a_n^2$ converges. Applying theorem II to the circle (C_n) with the segment $(a_n a_{n+1})$ as a diameter, it is shown very easily from (2) that each cut of the circle has the index zero. Therefore, there is one and only one zero of $f'(z)$ in (C_n), and it is, of course, on the real axis.

This method can be extended to the function

$$F(z) = e^{\alpha i z} f(z)$$

where $f(z)$ is given by (4) and α is a real number. If $|\alpha| < 1/\lambda_n$ where $2\lambda_n = |a_{n+1}| - |a_n|$, there is still one and only one zero of $F'(z)$ in the circle (C_n).

8. It is possible to extend Laguerre's theorem to functions of genus zero with a distribution of zeros of a much greater generality.

Let

$$f(z) = \Pi\left(1 - \frac{z}{a_n}\right)$$

where a_n are complex numbers, with modulus increasing with n, such that $\sum 1/|a_n|$ converges; further, we suppose that no two zeros are equal.

Let

$$|a_n| = n^{\sigma(n)}$$

where $\sigma(x)$ can be chosen as a twice differentiable function of x; further, let

$$\varphi(x) = \frac{d}{dx}[\sigma(x)\log x] = \frac{\sigma(x)}{x} + \sigma'(x)\log x > 0.$$

As the function is of genus zero, we have

$$\varliminf_{x \to \infty} \sigma(x) > 1.$$

We are going to consider separately two classes of functions of genus zero:

I. Let us suppose first that $x^{1/2}\varphi(x)$ tends to $+\infty$ with x, and, further, that $x\varphi(x)$ is an increasing function for sufficiently large values of x.

Let us assume the following hypotheses on the distribution of zeros in the plane:

A. If $$\alpha_n = \left(\widehat{\overrightarrow{Oa_n},\ \overrightarrow{a_na_{n+1}}}\right), \qquad |\alpha_n| \leqslant \alpha_0 < \frac{\pi}{2},$$
α_0 being a fixed angle.

B. Let D and D' be the straight lines drawn at right angles with the segment a_na_{n+1} from the points a_n and a_{n+1}; there are at most N zeros of $f(z)$ between D and D', N being independent of n.

C. There is no zero of $f(z)$ in the circle (C_n) with $a_n\, a_{n+1}$ as a diameter.

Then, there is one and only one zero of $f'(z)$ in each circle (C_n) for $n > n_0$, n_0 being a sufficiently large number. Further, if b_n is that zero, $|a_{n+1} - b_n| / |a_{n+1} - a_n|$ tends to zero with $1/n$.

The proof follows from theorem I, which we apply by showing that in the circle (C_n), $X > 0$, when $n > n_0$.

II. Suppose that $x^{1/2}\varphi(x)$ tends to zero with $1/x$, and that, either $x\varphi(x)$ is an increasing function of x, tending to $+\infty$ with x (in the case of functions $f(z)$ of order zero), or $1 < a \leqslant x\,\varphi(x) \leqslant a'$, where a and a' are fixed numbers (in the case of functions $f(z)$ of positive order $\rho < 1$).

Further, let us assume that:

A. $$\varphi\left(x + \frac{A}{\varphi(x)}\right) \geqslant B\,\varphi(x) \qquad \begin{bmatrix} 0 < B \leqslant 1 \\ A > 0 \end{bmatrix}$$

whatever the constant A may be, for all $x > x_0$; (x_0 and B may depend on A.

B. $$\varphi\left(x \pm \frac{\gamma}{\varphi(x)}\right) \geqslant h\,\varphi(x)$$

where $0 < \gamma \leq c$ and $x > x_0$, c being a fixed number, and $h > h_0$ for functions of order zero, $h > h_0^{1/2}$ for functions of positive order. Here h_0 is the root of the equation

$$\psi\left(1 + \frac{1}{h_0}\right) - \psi(1) = 2h_0$$

where

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The value of h_0 is $h_0 = 0.79\ldots$

We then make the following hypotheses on the distribution of the zeros:

C. $|\alpha_n| \leqslant \alpha_0 < \pi/2$, α_n having the same meaning as above.

D. Let (S_n) and (S'_n) be the circles tangent to the segment a_na_{n+1}

at its middle point A_n, and having a common radius

$$R_n = Kn\varphi(n).OA_n,$$

K being a conveniently chosen number. We assume that all the zeros of $f(z)$ are outside these circles.

E. There are no zeros of $f(z)$ in the circle (C_n).

Then there is one, and only one zero of $f'(z)$ in the circle (C_n) for $n > n_0$, n_0 being a sufficiently large number.

The proof follows again from theorem I, by showing that in the circle (C_n), $|Y| < 1/\lambda_n$ when $n > n_0$.

It must be noted in this case that there is a necessary geometrical relation between α_n and R_n; it is easy to see that

$$\sin |\alpha_n| < \frac{1}{Kn\varphi(n)}.$$

For the functions of order zero such that $a < x^{1/2}\varphi(x) < b$, where a and b are fixed numbers, neither of the above methods applies; it is then necessary to make further hypotheses on the zeros.

DUALITY RELATIONS IN TOPOLOGY

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In some recent papers[1] I have introduced the relative cycles for point sets and the associated relative boundary relations and homologies which may be of four types: absolute, modular, relative, relative modular. Various considerations lead one also to introduce a couple of invariants analogous to the Betti-numbers and for all these I have given loc. cit. proofs of some very general relations, in particular of duality which include all those previously known.

In going over the whole question I have recently had occasion to revise the proofs and extend the results somewhat. The extensions are along the line of much information concerning the relative torsion coefficients which occur, however, only when the subset G of the carrying complex is polyhedral. I do not wish to dwell on these here. The modified proofs are noteworthy and the changes shall now be indicated in outline. Their object was to extend as far as possible Poincaré's own proof for the duality relations of an M_n without boundary and to avoid wherever possible